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Varieties of Anticommutative n-ary Algebras

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A fundamental problem in the theory of n-ary algebras is to determine the correct generalization of the Jacobi identity. This paper describes some computational results on this problem using representations of the symmetric group. It is well known that over a field of characteristic 0 any variety of n-ary algebras can be defined by multilinear identities. In the anticommutative case, it is shown that for $n \leq 8$ the $\binom{2n-1}{n}$-dimensional S_{2n-1}-module of multilinear identities in which each term involves two n-ary products (i.e., two pairs of n-ary anticommutative brackets) decomposes as the direct sum of the n distinct simple modules labelled by the n partitions of $2n - 1$ in which only 1 and 2 occur as parts. In the cases $n = 3$ (resp. $n = 4$), the kernel of the commutator expansion map and a generator for each of the 7 (resp. 15) nonzero submodules are determined. The paper concludes with some conjectures for $n \geq 5$. © 1997 Academic Press

INTRODUCTION

The general theory of n-ary algebras has been studied since the late 1960s, beginning with fundamental work by Russian mathematicians. Some of this work can be found in the paper of Kurosh [Ku], and other papers in the same volume. For a survey, dating from the mid-1970s, of the achievements of the Russian school in this area, see the paper of Baranovich and Burgin [BB] and the references therein.

One of the most interesting problems in the theory of n-ary algebras is to find the "correct" generalization of the notion of Lie algebra to the n-ary case. Some early results on this problem are mentioned in [Ku, Sect. 6] and [BB, Sect. 15], but substantial progress was not made until the

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work of Filippov [F] in the mid-1980s, followed 10 years later by the papers of Hanlon and Wachs [HW] and Gnedbaye [G].

All of the above-mentioned papers make the natural assumption that a Lie n-ary algebra should be anticommutative in the n-ary sense: when two factors in the n-ary product are equal, the product is zero. (For example, any alternative (binary) algebra under the ternary product $[a, b, c] := (ab)c - a(bc)$ (the associator) is an anticommutative ternary algebra.) The interesting part of the problem is to find the n-ary generalization of the Jacobi identity.

The discussion of Lie n-algebras in [Ku] and [BB] focuses on the problem of determining identities in the free anticommutative n-algebra which become zero when the brackets are expanded as commutators in the free associative n-algebra. (In the binary case, the commutator expansion is $[a, b] \mapsto ab - ba$. The general n-ary definition is given immediately before Proposition 1 below.) A more general approach to the problem can be based on the fact, well-known in the theory of identities of algebras, that over any field of characteristic zero, the varieties of n-algebras defined by identities in which each term involves k occurrences of the n-ary operation are in bijective correspondence with the S_d-submodules of the space of all multilinear polynomials in which each term involves k occurrences of the n-ary operation, where d is the degree of these (homogeneous) polynomials. Here we may replace "n-algebras" by "anticommutative n-algebras" (or any other variety) and the result still holds. (An example of classifying varieties of Lie algebras by this method may be found in [B, Sect. 4.8.3].)

In this paper we describe some results in the cases $n = 3, 4$, $k = 2$, giving

(1) the decomposition into simple submodules of the "anticommutative" S_d-module;

(2) the kernel of the commutator expansion map;

(3) an explicit generator for each submodule; many of these identities can be regarded as generalizations of the Jacobi identity.

The paper concludes with some conjectures for $n \geq 5$.

Most of the calculations in what follows were done using Maple V.3 on a Sun Sparcstation LX.

DEFINITIONS AND ELEMENTARY RESULTS

Let A be a vector space over a field F, let n be an integer ≥ 2, and let $\omega: A^n \to A$ be a multilinear map. We write $a_1 \cdots a_n$ instead of

$\omega(a_1, \ldots, a_n)$ and call A an *n-ary algebra*, or simply an *n-algebra*. Given integers $r \geq 1$ and $n \geq r + 2$, and an n-algebra A, we can fix r (ordered) elements $b_1, \ldots, b_r$ of A and define an $(n - r)$-algebra structure on the vector space A by $a_1 \cdots a_{n-r} = a_1 \cdots a_{n-r} b_1 \cdots b_r$; we call this the *$r$-reduced* $(n - r)$-structure on A corresponding to $b_1, \ldots, b_r$. By a *derivation* of an n-algebra A we mean an endomorphism d of A satisfying $d(a_1 \cdots a_n) = \sum_{i=1}^{n} a_1 \cdots d(a_i) \cdots a_n$. If $a_1 \cdots (a_i \cdots a_{i+n-1}) \cdots a_{2n-1} = a_1 \cdots (a_j \cdots a_{j+n-1}) \cdots a_{2n-1}$ for all $1 \leq i, j \leq n$ we call A an *associative* n-algebra. If $a_1 \cdots a_n = 0$ whenever $a_i = a_j$ for some $i \neq j$, we call A an *anti-commutative* n-algebra; in this case we write $[a_1 \cdots a_n]$ instead of $a_1 \cdots a_n$. If $\operatorname{char} F \neq 2$ we have the equivalent condition $[a_{\pi(1)} \cdots a_{\pi(n)}] = \epsilon(\pi)[a_1 \cdots a_n]$ for any π in the symmetric group S_n where $\epsilon \colon S_n \to \{\pm 1\}$ is the sign homomorphism. Given $n - 1$ elements $a_1, \ldots, a_{n-1}$ of an anticommutative n-algebra A, we define the corresponding *adjoint mapping* $\operatorname{ad}_{a_1, \ldots, a_{n-1}} \colon A \to A$ by $\operatorname{ad}_{a_1, \ldots, a_{n-1}}(b) = [a_1 \cdots a_{n-1} b]$.

By an *identity* of anticommutative n-algebra A we mean a polynomial f in some free anticommutative n-algebra which is identically zero when the generators are replaced by any elements of A. If $\operatorname{char} F = 0$ then any identity is equivalent to the linearizations of its homogeneous components; if $\operatorname{char} F > d$ then any homogeneous identity of degree $\leq d$ is equivalent to its linearization. See [O, Sect. 3] for these results in the case $n = 2$, and [BB, Sect. 1] and the references therein for the general case.

Denote by P_n^k the space of multilinear identities for anticommutative n-algebras in which each term involves k pairs of n-ary anticommutative brackets. Then a basis for P_n^k consists of all multilinear anticommutative n-ary monomials of degree $kn - (k - 1)$. (We can think of each monomial as a rooted n-ary tree with k branch nodes. Such a tree has kn leaves, but we must subtract the $k - 1$ leaf nodes where the $k - 1$ nonroot branch nodes are attached.)

For $d = kn - (k - 1)$ the symmetric group S_d acts on the space P_n^k by permuting the d letters. If an n-algebra A satisfies the identity $f \in P_n^k$ then it also clearly satisfies $\pi . f$ for any $\pi \in S_d$, and hence it satisfies any linear combination of permuted forms of f. Thus it satisfies all the identities in the S_d-submodule $S(f)$ of P_n^k generated by f. From these remarks it follows that we can classify the varieties of anticommutative n-algebras defined by multilinear identities in which each term involves k pairs of n-ary anticommutative brackets by determining the S_d-submodules of P_n^k.

Since S_d sends any monomial into another monomial, with a change of sign in some cases, the matrices representing the action of S_d on P_n^k are signed permutation matrices. These matrices are orthogonal with respect to the inner product on P_n^k defined by declaring the monomials to be an orthonormal basis. We may therefore assume that distinct S_d-submodules of P_n^k are orthogonal subspaces.

The simple S_d-modules (and the corresponding characters) are indexed by the partitions of d. We give these partitions the standard order: $[d], [d-1, 1], [d-2, 2], [d-2, 1^2], \dots, [1^d]$. When $k = 2$ we have $d = 2n - 1$; in this case the last n partitions are just those which have only 1 and 2 as parts.

A (finite-dimensional) S_d-module V is *multiplicity-free* if each isotypic component is simple; i.e., each simple summand of V occurs exactly once. In this case V has only finitely many submodules: if we write $V = U_1 \oplus \cdots \oplus U_m$, where the U_i are distinct simple modules, then there are exactly 2^m submodules of V (each U_i is either in or out). The same result does not hold when an isotypic component is not simple: for example, if $V = U \oplus U$ with U simple then there are infinitely many different subspaces of V isomorphic to U as an S_d-module.

The space P_n^0 is one-dimensional, with basis a; this identity defines the variety consisting of the zero algebra alone. The space P_n^1 is also one-dimensional, with basis $[a_1 a_2 \cdots a_n]$; this identity defines the variety of trivial (or abelian) anticommutative n-algebras. The space P_n^2 has dimension $\binom{2n-1}{n}$, with basis consisting of the monomials

$$x = \big[[a_{\pi(1)} a_{\pi(2)} \cdots a_{\pi(n)}] a_{\pi(n+1)} a_{\pi(n+2)} \cdots a_{\pi(2n-1)}\big]$$

for some permutation $\pi \in S_{2n-1}$; by anticommutativity we may assume that $\pi(1) < \cdots < \pi(n)$ and $\pi(n+1) < \cdots < \pi(2n-1)$. We define the sign of the monomial x by $\epsilon(x) = \epsilon(\pi)$.

Although we will not need the result, it is easy to work out a general formula for $\dim P_n^k$. We first define the *expansion* of an anticommutative monomial by

$$[a_1 \cdots a_n] \mapsto \sum_{\pi \in S_n} \epsilon(\pi) a_{\pi(1)} \cdots a_{\pi(n)}.$$

Here the right-hand side is to be regarded as an element of the free associative n-algebra. Expanding all k commutators in each term of an element of P_n^k induces a linear map, the *commutator expansion map*, from P_n^k to the $d!$-dimensional space ($d = kn - (k-1)$) of multilinear associative n-ary polynomials in which each term involves k occurrences of the n-ary associative product. (Put more simply, the codomain of the commutator expansion map is the span of all associative words in d symbols.) It is clear that this map is a morphism of S_d-modules.

PROPOSITION 1. *The dimension of* P_n^k *is* $(kn)!/(k!(n!)^k)$.

Proof. When the brackets in each anticommutative monomial are expanded in the free nonassociative n-algebra, each of the k pairs of brackets multiplies the number of terms by $n!$. So the total number of terms in the expansion of one of the basis monomials of P_n^k is $(n!)^k$. When we expand all of the $\dim P_n^k$ monomials this way, each nonassociative monomial occurs as a term exactly once. The number of arrangements of letters in a nonassociative monomial is $(kn-(k-1))!$, and the number of different bracket arrangements is the n-Catalan number $(1/k)\binom{kn}{k-1}$ (see [K], p. 396, exercise 11). Therefore

$$(n!)^k \dim P_n^k = (kn-(k-1))!\,\frac{1}{k}\binom{kn}{k-1},$$

which gives the result. ∎

Here is a short table:

n	2	3	4	5	6	7	8	9
$\dim P_n^2$	3	10	35	126	462	1,716	6,435	24,310
$\dim P_n^3$	15	280	5,775	26,126	2,858,856	66,512,160	1,577,585,295	37,978,905,250

The remainder of this paper is concerned with identities in which each term involves two pairs of anticommutative brackets ($k=2$). In the familiar case $n=2$ we have the S_3-module P_2^2 with basis $[[ab]c]$, $[[ac]b]$, $[[bc]a]$. One easily checks that P_2^2 decomposes as the direct sum of two simple submodules:

(i) the first spanned by the two equivalent identities $[[ab]c]+[[ac]b]$ and $[[ac]b]+[[bc]a]$ (this subspace affords the two-dimensional simple S_3-module);

(ii) the second spanned by the Jacobi identity $[[ab]c]-[[ac]b]+[[bc]a]$ (this subspace affords the signature character of S_3).

The identity $[[ac]b]+[[bc]a]$ is equivalent (assuming $\operatorname{char} F \neq 2$) to the identity $[a[ab]]$, which can be expressed as $(\mathrm{ad}_a)^2=0$. This is called the *2-Engel* identity; some results on anticommutative algebras satisfying an Engel condition may be found in [Kuz].

The Jacobi identity may be characterized in at least four different ways, which lead to different identities in the n-ary case:

(1) It is the kernel of the commutator expansion map; n-ary versions of this property are discussed in [Ku, Sect. 6] and [BB, Sect. 15].

(2) It expresses the derivation property of the adjoint mapping; n-ary versions of this are discussed in [F], [HW], and [G].

(3) It is the sum of the monomials over the powers of a cyclic permutation of the letters.

(4) It spans a one-dimensional space affording the signature character of the symmetric group.

We will see n-ary generalizations of these properties in what follows. The results of this paper can be described as a representation theoretic analysis of the relations between these n-ary generalizations in the cases $n = 3, 4$.

THE CASE $n = 3$: TERNARY ALGEBRAS

The S_5-module P_3^2 has a basis consisting of the 10 monomials

$$x_0 = [[abc]de], \qquad x_1 = [[abd]ce], \qquad x_2 = [[abe]cd],$$
$$x_3 = [[acd]be], \qquad x_4 = [[ace]bd], \qquad x_5 = [[ade]bc],$$
$$x_6 = [[bcd]ae], \qquad x_7 = [[bce]ad], \qquad x_8 = [[bde]ac],$$
$$x_9 = [[cde]ab].$$

These monomials have signs

i	0	1	2	3	4	5	6	7	8	9
$\epsilon(x_i)$	+	−	+	+	−	+	−	+	−	+

To describe the submodules of P_3^2 we recall the last three rows of the character table of S_5 (see [JK, Appendix I]):

	—	2	22	3	32	4	5
	1	10	15	20	20	30	24
$[2^2 1]$	5	-1	1	-1	-1	1	0
$[21^3]$	4	-2	0	1	1	0	-1
$[1^5]$	1	-1	1	1	-1	-1	1

The first row contains the cycle types (with 1 omitted from the partitions), the second row the class sizes, and the first column contains the partitions labelling the simple modules. The characters form an orthonormal basis for the space of all class functions on S_5 with respect to the usual inner product.

The first step in the classification of varieties of anticommutative ternary algebras is the following result.

PROPOSITION 2. *As S_5-modules we have $P_3^2 \cong [2^2 1] \oplus [21^3] \oplus [1^5]$.*

Proof. We choose a system of conjugacy class representatives and determine the action of each as a signed permutation of the ten monomials:

$$
\begin{array}{rcrrrrrrrrrr}
\mathrm{id} &=& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9, \\
(de) &=& -0 & 2 & 1 & 4 & 3 & -5 & 7 & 6 & -8 & -9, \\
(bc)(de) &=& 0 & 4 & 3 & 2 & 1 & 5 & -7 & -6 & -9 & -8, \\
(cde) &=& -1 & -2 & 0 & 5 & -3 & -4 & 8 & -6 & -7 & 9, \\
(ab)(cde) &=& 1 & 2 & -0 & 8 & -6 & -7 & 5 & -3 & -4 & -9, \\
(bcde) &=& -3 & -4 & -0 & -5 & -1 & -2 & 9 & 6 & 7 & 8, \\
(abcde) &=& -6 & -7 & 0 & -8 & 1 & 2 & -9 & 3 & 4 & 5,
\end{array}
$$

From this we see that the traces of the representatives are $(10, -4, 2, 1, -1, 0, 0)$. This is the sum of the last three rows of the character table. ∎

For convenience we abbreviate the three simple submodules of P_3^2 by their dimensions, writing $[5], [4], [1]$ for $[2^2 1], [21^3], [1^5]$, respectively. Since P_3^2 is multiplicity-free there are exactly eight submodules: P_3^2 itself, $[5] \oplus [4]$, $[5] \oplus [1]$, $[5]$, $[4] \oplus [1]$, $[4]$, $[1]$, and $\{0\}$. Any variety of anticommutative ternary algebras, defined by multilinear identities in which each term involves two pairs of brackets, corresponds to one of the seven nonzero submodules in this list.

THEOREM 1. *The kernel of the commutator expansion map from P_3^2 to the* 120-*dimensional span of all associative words in five symbols is* $\{0\}$.

Proof. To compute the kernel, one constructs the 120×10 matrix in which the columns are labelled by the basis monomials $x_0, \ldots, x_9$ and each column contains the coefficients of the expansion of the corresponding monomial with respect to the basis of all associative words in five symbols. One then determines that this matrix has rank 10, so the result follows. ∎

Theorem 1 is mentioned without proof in [Ku] and [BB]; it shows that in the case $n = 3$ there is no identity generalizing the commutator expansion property of the Jacobi identity.

Given any element $f \in P_3^2$ expressed as a row vector $v \in F^{10}$ with respect to the basis $x_0, \ldots, x_9$, we can apply every $\pi \in S_5$ to v to obtain a 120×10 matrix over F. We can then find a subset of the rows which forms a basis of the row space by initializing $R = \{v\}$, $r = 1$, and then repeating the following steps until $r = 120$: add 1 to r; determine whether row r is in span(R); if not, then add row r to R. We use the lexicographical ordering on the elements of S_5: writing $\pi = \pi(a)\pi(b)\pi(c)\pi(d)\pi(e)$ we have $\pi_1 = abcde$, $\pi_2 = abced, \ldots, \pi_{120} = edcba$. In this way we can identify a basis of the submodule $S(f)$ with a subset of $\{1, 2, \ldots, 120\}$.

THEOREM 2. *Let F be a field of characteristic* 0 (*resp.* > 5). *There are exactly seven varieties of anticommutative ternary algebras over F defined by identities* (*resp. homogeneous identities*) *in which each term involves two pairs of anticommutative brackets. These varieties are defined by the following seven identities, which are listed together with the corresponding submodules of P_3^2*:

$$
\begin{array}{ll}
P_3^2, & I_{541} = [[abc]de], \\
[5] \oplus [4], & I_{54} = [[abd]cd], \\
[5] \oplus [1], & I_{51} = [[abc]de] + [[bcd]ea] + [[cde]ab] + [[dea]bc] \\
& \qquad +[[eab]cd], \\
[5], & I_5 = [ab[abc]], \\
[4] \oplus [1], & I_{41} = [ab[cde]] - [[abc]de] - [c[abd]e] - [cd[abe]], \\
[4], & I_4 = [[abd]cd] + [[bcd]ad] + [[cad]bd], \\
[1], & I_1 = \sum_{i=0}^{9} \epsilon(x_i)x_i.
\end{array}
$$

Proof. It is clear that any monomial x_i generates P_3^2, so we can take $I_{541} = x_0$ as the generator.

If we linearize I_{54} by replacing d by $d + e$ we obtain $[[abd]ce] + [[abe]cd]$. This element of P_3^2 generates a nine-dimensional submodule with basis $1, 3, 7, 9, 13, 31, 33, 37, 61$. The only nine-dimensional submodule of P_3^2 is $[5] \oplus [4]$. The identity I_{54} is equivalent to $[[ab]c]$ in the reduced algebra corresponding to d.

Normalizing the monomials in I_{51} using anticommutativity we obtain

$$[[abc]de] + [[abe]cd] + [[ade]bc] - [[bcd]ae] + [[cde]ab].$$

This element of P_3^2 generates a six-dimensional submodule with basis $1, 2, 3, 4, 5, 7$. The only six-dimensional submodule of P_3^2 is $[5] \oplus [1]$.

If we linearize I_5 by replacing a, b by $a + d, b + e$ and then normalize the monomials we obtain

$$I_5' = [[abc]de] + [[ace]bd] + [[bcd]ae] + [[cde]ab].$$

This generates a five-dimensional submodule with basis $1, 2, 3, 4, 5$. This submodule could be either $[5]$ or $[4] \oplus [1]$. The trace of a transposition on this submodule is -1, so this submodule must be $[5]$.

Normalizing I_{41} we obtain

$$-[[abc]de] + [[abd]ce] - [[abe]cd] + [[cde]ab].$$

This generates a five-dimensional submodule with basis $1, 7, 13, 19, 31$. The trace of a transposition on this submodule is -3, so this submodule must be $[4] \oplus [1]$.

If we linearize I_4 by replacing d by $d + e$ and then normalize the monomials we obtain

$$I_4' = [[abd]ce] + [[abe]cd] - [[acd]be] - [[ace]bd] + [[bcd]ae] + [[bce]ac].$$

This generates a four-dimensional submodule with basis $1, 3, 9, 33$; this submodule can only be [4].

It is easy to see that $\pi.I_1 = \epsilon(\pi)I_1$ for any $\pi \in S_5$, so I_1 spans [1]. ∎

The identity I_5 defines the variety of anticommutative ternary algebras satisfying the 2-Engel identity $(\mathrm{ad}_{ab})^2 = 0$. The identity I_{41} expresses the property that ad_{ab} is a derivation of the ternary product; this is seen more clearly if we write I_{41} in the form

$$[ab[cde]] = [[abc]de] + [c[abd]e] + [cd[abe]].$$

COROLLARY 1. *The S_5-module P_3^2 is the direct sum of the submodule generated by the 2-Engel identity I_5 and the submodule generated by the derivation identity I_{41}.*

These results provide four ternary generalizations of the Jacobi identity:

(1) The identity I_{41} generalizes the derivation property of the Jacobi identity.

(2) The identity I_{51} generalizes the cyclic property of the Jacobi identity.

(3) The identity I_1 spans [1]; so the variety defined by I_1 generalizes the signature character property of the Jacobi identity.

(4) The identity I_4 expresses the property that every reduced algebra (corresponding to d) satisfies the Jacobi identity.

Three of these identities are closely related: since $S(I_{41}) = S(I_4') \oplus S(I_1)$, the identity I_{41} implies each of I_4 and I_1, and is implied by the pair of them. However, $S(I_{51})$ intersects $S(I_{41})$ only in [1] and is the complementary submodule to $S(I_4')$.

THE CASE $n = 4$

The S_7-module P_4^2 has a basis consisting of the 35 monomials

$$
\begin{array}{lll}
y_0 = [[abcd]efg], & y_1 = [[abce]dfg], & y_2 = [[abcf]deg], \\
y_3 = [[abcg]def], & y_4 = [[abde]cfg], & y_5 = [[abdf]ceg], \\
y_6 = [[abdg]cef], & y_7 = [[abef]cdg], & y_8 = [[abeg]cdf], \\
y_9 = [[abfg]cde], & y_{10} = [[acde]bfg], & y_{11} = [[acdf]beg], \\
y_{12} = [[acdg]bef], & y_{13} = [[acef]bdg], & y_{14} = [[aceg]bdf], \\
y_{15} = [[acfg]bde], & y_{16} = [[adef]bcg], & y_{17} = [[adeg]bcf], \\
y_{18} = [[adfg]bce], & y_{19} = [[aefg]bcd], & y_{20} = [[bcde]afg], \\
y_{21} = [[bcdf]aeg], & y_{22} = [[bcdg]aef], & y_{23} = [[bcef]adg], \\
y_{24} = [[bceg]adf], & y_{25} = [[bcfg]ade], & y_{26} = [[bdef]acg], \\
y_{27} = [[bdeg]acf], & y_{28} = [[bdfg]ace], & y_{29} = [[befg]acd], \\
y_{30} = [[cdef]abg], & y_{31} = [[cdeg]abf], & y_{32} = [[cdfg]abe], \\
y_{33} = [[cefg]abd], & y_{34} = [[defg]abc]. &
\end{array}
$$

The sequence of 35 signs of these monomials is

$$+-+-+-++-+-+--+-+-+-+-++-+-+-++-+-+.$$

The last four rows of the character table of S_7 are given in Table 1.

PROPOSITION 3. *As S_7-modules we have* $P_4^2 \cong [2^3 1] \oplus [2^2 1^3] \oplus [21^5] \oplus [1^7]$.

Proof. As in Proposition 2, we choose a set of conjugacy class representatives and compute the action of each as a signed permutation of the 35 monomials. This gives the traces of the representatives: $(35, -15, 7, -3, 5, -3, 1, 2, -1, 1, -1, 0, 0, 0, 0)$. This is the sum of the last four rows of the character table. ∎

We denote the simple modules corresponding to the last four partitions of 7 by [14], [14′], [6], and [1].

THEOREM 3. *The kernel of the commutator expansion map from P_4^2 to the* 5040-*dimensional span of all associative words in seven symbols is the submodule* [1].

Proof. As in Theorem 1, one constructs the 5040×35 matrix in which the columns are labelled by the basis monomials $y_0, \ldots, y_{34}$ and each column contains the coefficients of the expansion of the corresponding monomial with respect to the basis of all associative words in seven symbols. One then determines that this matrix has rank 34, so the kernel of the commutator expansion map has dimension 1. Since the map is an S_7-module morphism, the kernel is an S_7-submodule of P_4^2, and so the result follows. ∎

Theorem 3 shows that identity I_1 in the following result generalizes the commutator expansion property of the Jacobi identity.

THEOREM 4. *Let F be a field of characteristic* 0 (*resp.* > 7). *There are exactly* 15 *varieties of anticommutative* 4-*ary algebras over F defined by identities* (*resp. homogeneous identities*) *in which each term involves two pairs*

TABLE 1

	—	2	22	222	3	32	322	33	4	42	43	5	52	6	7
	1	21	105	105	70	420	210	280	210	630	420	504	504	840	720
$[2^3 1]$	14	-4	2	0	-1	-1	-1	2	2	0	-1	-1	1	0	0
$[2^2 1^3]$	14	-6	2	-2	2	0	2	-1	0	0	0	-1	-1	1	0
$[21^5]$	6	-4	2	0	3	-1	-1	0	-2	0	1	1	1	0	-1
$[1^7]$	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	-1	1

of anticommutative brackets. These varieties are defined by the following 15 *identities, which are listed together with the corresponding submodules of* P_4^2; $L(I)$ *denotes the multilinearization of the identity* I.

$$
\begin{array}{ll}
P_4^2, & I_{35} = [[abcd]efg], \\
[14] \oplus [14'] \oplus [6], & I_{34} = [[abcf]def], \\
[14] \oplus [14'] \oplus [1], & I_{29} = [[abcd]efg] + [[bcde]fga] + [[cdef]gab] \\
 & \qquad +[[defg]abc] + [[efga]bcd] + [[fgab]cde] \\
 & \qquad +[[gabc]def], \\
[14] \oplus [14'], & I_{28} = [[abde]cde], \\
[14] \oplus [6] \oplus [1], & I_{21} = [[abcd]efg] - [[abce]dfg] + [[abcf]deg] \\
 & \qquad -[[abcg]def] - 2[[defg]abc], \\
[14'] \oplus [6] \oplus [1], & I_{21'} = [abc[defg]] - [[abcd]efg] - [d[abce]fg] \\
 & \qquad -[de[abcf]g] - [def[abcg]], \\
[14] \oplus [6], & I_{20} = [[abcf]def] + [[bcdf]eaf] + [[cdef]abf] \\
 & \qquad +[[deaf]bcf] + [[eabf]cdf], \\
[14'] \oplus [6], & I_{20'} = [ab[cdef]f] - [[abcf]def] - [c[abdf]ef] \\
 & \qquad -[cd[abef]f], \\
[14] \oplus [1], & I_{15} = [[abcd]efg] + [[abfg]cde] - [[acef]bdg] \\
 & \qquad -[[adeg]bcf] - [[bceg]adf] - [[bdef]acg] \\
 & \qquad -[[cdfg]abe], \\
[14'] \oplus [1], & I_{15'} = L(I_{14'}) + I_1, \\
[14], & I_{14} = [abc[abcd]], \\
[14'], & I_{14'} = [a[bcde]de] - [[abde]cde] - [b[acde]de], \\
[6] \oplus [1], & I_7 = \sum_{i=0}^{19} \epsilon(y_i) y_i, \\
[6], & I_6 = \sum_{i=0}^{9} \epsilon(x_i)[[x_{i1}x_{i2}x_{i3}f]x_{i4}x_{i5}f], \\
[1], & I_1 = \sum_{i=0}^{34} \epsilon(y_i) y_i.
\end{array}
$$

Proof. The proof is similar to the proof of Theorem 2, with the following exceptions:

(1) The identities I_{21} and I_{15} were determined by finding generators for the orthogonal complements of the submodules generated by the multilinearizations of I_{14} and I_{20} (respectively).

(2) Since P_4^2 is multiplicity-free, any submodule is generated by the sum of generators of its simple components; this explains identity $I_{15'}$. A more compact generator was not found for the submodule $[14'] \oplus [1]$. ∎

The identity I_{14} expresses the 2-Engel condition for 4-ary algebras: $(\mathrm{ad}_{abc})^2 = 0$. The identity $I_{21'}$ says that every adjoint map ad_{abc} is a derivation of the 4-ary product.

COROLLARY 2. *The S_7-module P_4^2 is the direct sum of the submodule generated by the 2-Engel identity I_{14} and the submodule generated by the derivation identity $I_{21'}$.*

From these results we find eight 4-ary generalizations of the Jacobi identity:

(1) Identity $I_{21'}$ generalizes the derivation property.

(2) Identity $I_{20'}$ says that the reduced algebra corresponding to f satisfies the ternary derivation property.

(3) Identity $I_{14'}$ says that the 2-reduced algebra corresponding to d, e satisfies the Jacobi identity.

(4) Identity I_{29} generalizes the cyclic property.

(5) Identity I_{20} says that the reduced algebra corresponding to f satisfies the ternary cyclic property.

(6) Identity I_1 generalizes the signature property, and (by Theorem 3) also generalizes the commutator expansion property.

(7) Identity I_7 is similar to I_1, but involves a sum only over the monomials which begin with the letter a.

(8) Identity I_6 says that the reduced algebra corresponding to f satisfies the ternary signature property.

There does not seem to be any obvious description of identities I_{21}, I_{15}, and $I_{15'}$ as generalizations of the Jacobi identity. From the decompositions we see that $I_{21'}$ implies $I_{20'}$, $I_{14'}$, I_6, and I_1, so these five generalizations of the Jacobi identity are closely related. Other generalizations are provided by I_{29} and I_{20}. As in the ternary case, this representation theoretic viewpoint suggests that the derivation identity $I_{21'}$ is the most natural choice for the 4-ary generalization of the Jacobi identity.

CONJECTURES FOR THE GENERAL CASE

We conclude with some conjectures based on the computational data presented in the previous sections.

Conjecture 1. Let P_n^2 denote the $\binom{2n-1}{n}$-dimensional S_{2n-1}-module spanned by the n-ary anticommutative monomials involving two pairs of brackets. Then P_n^2 is isomorphic to the direct sum of the simple modules labelled by the partitions $[2^{n-1}1], [2^{n-2}1^3], \ldots, [1^{2n-1}]$ (the n partitions of $2n-1$ which have only 1 and 2 as parts), and is therefore multiplicity-free.

It follows that, for every $n \geq 2$, there are exactly $2^n - 1$ distinct varieties of anticommutative n-algebras defined by identities in which each term involves two pairs of brackets.

Conjecture 1 generalizes Propositions 2 and 3. Computer calculations have verified this conjecture for $n \leq 8$.

Conjecture 2. The kernel of the commutator expansion map from P_n^2 to the S_{2n-1}-module of associative words on $2n - 1$ symbols has dimension 1 for n even and 0 for n odd. For n even, the kernel is isomorphic to the simple module labelled by the partition $[1^{2n-1}]$.

Conjecture 2 generalizes Theorems 1 and 3.

Conjecture 3. The module P_n^2 is the direct sum of the submodule generated by the (linearization of the) 2-Engel identity and the submodule generated by the derivation identity. The former is isomorphic to the simple module labelled by the partition $[2^{n-1}1]$, and the latter is isomorphic to the direct sum of the simple modules labelled by the partitions $[2^{n-2}1^3], \ldots, [1^{2n-1}]$.

Conjecture 3 generalizes Corollaries 1 and 2.

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